ON A COMPLETE SET OF ORTHOGONAL F-SQUARES OF ORDER 8
WITH A MATELESS LATIN SQUARE

by

W. T. Federer, J. P. Mandeli, and S. J. Schwager

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Abstract

A Latin square of order eight that has no orthogonal Latin square is shown to have 42 mutually orthogonal $F(8;4,4)$-square mates. This is the complete set of F-squares needed for the mateless Latin square. A new F-square geometry of order eight and a new orthogonal array involving three sets of eight symbols and 42 sets of two symbols are then constructed.

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Mandeli (1975) and Mandeli and Federer (1981) showed how to construct six F-squares, denoted as F(4;2,2)-squares, that are mutually orthogonal and are orthogonal to a Latin square of order four that has no orthogonal Latin square mate, i.e., an [OL(4,1)] set. The method of construction follows. The [OL(4,1)]-square

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

is decomposed into three orthogonal F-squares, F(4;2,2), as follows:

$$F_1 = \begin{bmatrix} + & + & - & - \\ - & + & + & - \\ - & - & + & + \\ + & - & - & + \end{bmatrix}, \quad F_2 = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}, \quad \text{and} \quad F_3 = \begin{bmatrix} + & - & - & + \\ + & + & - & - \\ - & + & + & - \\ - & - & + & + \end{bmatrix} .$$

Then, given

$$H = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix} ,$$

six orthogonal F-squares, OF(4;2,2;6), are constructed as follows:

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$F_4 = F_1 \times$ row 3 of H $= F_3 \times$ column 3 of H ,

$F_5 = F_2 \times$ row 2 of H ,

$F_6 = F_2 \times$ row 4 of H ,

$F_7 = F_3 \times$ row 3 of H $= F_1 \times$ column 3 of H ,

$F_8 = F_2 \times$ column 2 of H , and

$F_9 = F_2 \times$ column 4 of H .

To illustrate the procedure, the F-square F_4 is obtained by multiplying each row of F_1 by row 3 of H:

$$\begin{array}{c|cccc} \text{Row 3 of H} & + & - & + & - \\ \hline & + & + & - & - \\ F_1 & - & + & + & - \\ & - & - & + & + \\ & + & - & - & + \end{array} \quad \rightarrow \quad F_4 = \begin{bmatrix} + & - & - & + \\ - & - & + & + \\ - & + & + & - \\ + & + & - & - \end{bmatrix} ,$$

and the F-square F_8 is obtained by multiplying each column of F_2 by column 2 of H:

$$\begin{array}{c|cccc} \text{Column 2 of H} & & F_2 & & \\ \hline + & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ - & - & + & - & + \end{array} \quad \rightarrow \quad F_8 = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ - & + & - & + \\ + & - & + & - \end{bmatrix} .$$

Note that $F_1 \times$ row 2 of H does not produce an F-square:

$$\begin{array}{c|cccc} \text{Row 2 of H} & + & + & - & - \\ \hline & + & + & - & - \\ F_1 & - & + & + & - \\ & - & - & + & + \\ & + & - & - & + \end{array} \quad \rightarrow \quad \begin{bmatrix} + & + & + & + \\ - & + & - & + \\ - & - & - & - \\ + & - & + & - \end{bmatrix} \neq \text{F-square} .$$

Also note that there are $3 \times 3 = 9$ squares produced by row operations and $3 \times 3 = 9$ squares produced by column operations. Of these, eight are F-squares and ten are not F-squares.

The question now arises as to whether or not the same operations produce $(8-2)(8-1) = 42$ F-squares of order 8 that are mutually orthogonal and are orthogonal to a mateless Latin square of order 8, i.e., an [OL(8,1)]-set. We first decompose the following mateless Latin square into seven mutually orthogonal $F(8;4,4)$-squares, F_1 to F_7. These together with a Hadamard matrix H of order eight are given below:

LS(8)

```
1 2 3 4 5 6 7 8
8 1 2 3 4 5 6 7
7 8 1 2 3 4 5 6
6 7 8 1 2 3 4 5
5 6 7 8 1 2 3 4
4 5 6 7 8 1 2 3
3 4 5 6 7 8 1 2
2 3 4 5 6 7 8 1
```

F_1

```
+ + + + - - - -
- + + + + - - -
- - + + + + - -
- - - + + + + -
- - - - + + + +
+ - - - - + + +
+ + - - - - + +
+ + + - - - - +
```

F_2

```
+ + - - + + - -
- + + - - + + -
- - + + - - + +
+ - - + + - - +
+ + - - + + - -
- + + - - + + -
- - + + - - + +
+ - - + + - - +
```

F_3

```
+ + - - - - + +
+ + + - - - - +
+ + + + - - - -
- + + + + - - -
- - + + + + - -
- - - + + + + -
- - - - + + + +
+ - - - - + + +
```

F_4

```
+ - + - + - + -
- + - + - + - +
+ - + - + - + -
- + - + - + - +
+ - + - + - + -
- + - + - + - +
+ - + - + - + -
- + - + - + - +
```

F_5

```
+ - + - - + - +
+ + - + - - + -
- + + - + - - +
+ - + + - + - -
- + - + + - + -
- - + - + + - +
+ - - + - + + -
- + - - + - + +
```

F_6

```
+ - - + + - - +
+ + - - + + - -
- + + - - + + -
- - + + - - + +
+ - - + + - - +
+ + - - + + - -
- + + - - + + -
- - + + - - + +
```

F_7

```
+ - - + - + + -
- + - - + - + +
+ - + - - + - +
+ + - + - - + -
- + + - + - - +
+ - + + - + - -
- + - + + - + -
- - + - + + - +
```

H

```
+ + + + + + + +
+ - + - + - + -
+ + - - + + - -
+ - - + + - - +
+ + + + - - - -
+ - + - - + - +
+ + - - - - + +
+ - - + - + + -
```

Examining the $7 \times 7 = 49$ squares produced by the row operations described above, we obtain 28 orthogonal F-squares as follows:

$F_8 = F_1 \times$ row 2 of H $= F_5 \times$ column 2 of H ,

$F_9 = F_2 \times$ row 2 of H $= F_6 \times$ column 2 of H ,

$F_{10} = F_3 \times$ row 2 of H $= F_7 \times$ column 2 of H ,

$F_{11} = F_5 \times$ row 2 of H $= F_1 \times$ column 2 of H ,

$F_{12} = F_6 \times$ row 2 of H $= F_2 \times$ column 2 of H ,

$F_{13} = F_7 \times$ row 2 of H $= F_3 \times$ column 2 of H ,

$F_{14} = F_1 \times$ row 3 of H ,

$F_{15} = F_3 \times$ row 3 of H ,

$F_{16} = F_4 \times$ row 3 of H ,

$F_{17} = F_5 \times$ row 3 of H ,

$F_{18} = F_7 \times$ row 3 of H ,

$F_{19} = F_1 \times$ row 4 of H ,

$F_{20} = F_3 \times$ row 4 of H ,

$F_{21} = F_4 \times$ row 4 of H ,

$F_{22} = F_5 \times$ row 4 of H ,

$F_{23} = F_7 \times$ row 4 of H ,

$F_{24} = F_2 \times$ row 5 of H ,

$F_{25} = F_4 \times$ row 5 of H ,

$F_{26} = F_6 \times$ row 5 of H ,

$F_{27} = F_2 \times$ row 6 of H ,

$F_{28} = F_4 \times$ row 6 of H ,

$F_{29} = F_6 \times$ row 6 of H ,

$F_{30} = F_2 \times$ row 7 of H ,

$F_{31} = F_4 \times$ row 7 of H ,

$F_{32} = F_6 \times$ row 7 of H ,

$F_{33} = F_2 \times$ row 8 of H ,

$F_{34} = F_4 \times$ row 8 of H , and

$F_{35} = F_6 \times$ row 8 of H .

None of the other 21 squares obtained by row operations are F-squares; there are no duplications of F-squares, either.

Proceeding next to the 49 squares produced by column operations, we obtain the following 14 mutually orthogonal $F(8;4,4)$-squares:

$F_{36} = F_4 \times$ column 3 of H ,

$F_{37} = F_4 \times$ column 4 of H ,

$F_{38} = F_2 \times$ column 5 of H ,

$F_{39} = F_4 \times$ column 5 of H ,

$F_{40} = F_6 \times$ column 5 of H ,

$F_{41} = F_2 \times$ column 6 of H ,

$F_{42} = F_4 \times$ column 6 of H ,

$F_{43} = F_6 \times$ column 6 of H ,

$F_{44} = F_2 \times$ column 7 of H ,

$F_{45} = F_4 \times$ column 7 of H ,

$F_{46} = F_6 \times$ column 7 of H ,

$F_{47} = F_2 \times$ column 8 of H ,

$F_{48} = F_4 \times$ column 8 of H , and

$F_{49} = F_6 \times$ column 8 of H .

When added to the orthogonal F-squares F_1 to F_{35}, these 14 additional F-squares give the complete set. Of the other 35 squares obtained by column operations, 21 are not F-squares, eight are F-squares but are not orthogonal to one or more of F_1 to F_{35}, and six are duplicates of squares in the first set, namely F_8 to F_{13} .

A complete set of 49 orthogonal F(8;4,4)-squares has been obtained, seven, F_1 to F_7, of which can be used to compose a mateless Latin square of order eight. Thus, for orders 4 and 8, a complete set has been obtained. The question arises as to whether or not this F-square geometry and the construction method holds for all 4t or only for $4t = 2^n$. Using an extension of the computer program at the end of this paper, one could check for F-squares of order 12 and 16. This is being done.

A proof that a complete set exists for F-squares of order 2^n should be possible. This is because of closure under multiplication of the elements in a Galois Field of 2^n elements. There are 2^{2n} effects, $2^n - 1$ are row effects and $2^n - 1$ are column effects. Without loss of generality, the same $H_{2^n \times 2^n}$ used for row and column operations can be used to decompose the mateless Latin square of order 2^n into $2^n - 1$ $F(2^n;2^{n-1},2^{n-1})$-squares. Note that the interaction of the rows and columns has $(2^n - 1)(2^n - 1)$ single degree of freedom contrasts and these may be used to construct the $(2^n - 1)(2^n - 2)$ $OF(2^n;2^{n-1},2^{n-1})$-squares that are orthogonal to a mateless Latin square of order 2^n . This means that a new Projective Geometry is available with one axis of dimension $2^n - 1$ and the other $(2^n - 1)(2^n - 2)$ axes of dimension one.

The computer program for finding the set of mutually orthogonal F(8;4,4)-squares consists of the following four APL functions:

```
      ∇ INITIALIZE8
[1]   ⍝ WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
[2]   ⍝ 8/22/81
[3]   ⍝ INITIALIZES FSQUARE MATRICES F[7×8×8] AND H[8×8]
[4]   ⍝ USES FN 'POSN8' TO SET DIAGONALS OF F[I;;] TO ¯1
[5]    F← 7 8 8 ρH← 8 8 ρ1
[6]    IMJ←I-J←⍉I←⍉ 8 8 ρι8
[7]    F[1;;]←H-2×POSN8 4 3 2 1 ¯4 ¯5 ¯6 ¯7
[8]    F[2;;]←H-2×POSN8 6 5 2 1 ¯2 ¯3 ¯6 ¯7
[9]    F[3;;]←H-2×POSN8 6 5 4 3 ¯2 ¯3 ¯4 ¯5
[10]   F[4;;]←H-2×POSN8 7 5 3 1 ¯1 ¯3 ¯5 ¯7
[11]   F[5;;]←H-2×POSN8 7 5 4 2 ¯1 ¯3 ¯4 ¯6
[12]   F[6;;]←H-2×POSN8 7 6 3 2 ¯1 ¯2 ¯5 ¯6
[13]   F[7;;]←H-2×POSN8 7 6 4 1 ¯1 ¯2 ¯4 ¯7
[14]   H[2; 2 4 6 8]←H[3; 3 4 7 8]←H[4; 2 3 6 7]←H[5; 5 6 7 8]←¯1
[15]   H[6; 2 4 5 7]←H[7; 3 4 5 6]←H[8; 2 3 5 8]←¯1
      ∇
```

```
      ∇ X←POSN8 V
[1]   ⍝ WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
[2]   ⍝ 8/22/81
[3]   ⍝ V IS A VECTOR OF INTEGERS
[4]   ⍝ INPUT MATRIX IMJ[8×8] COMES FROM 'INITIALIZE8'
[5]    X←+/[1](((⍴V),8,8)⍴IMJ)= 3 2 1 ⍉(8,8,⍴V)⍴V
      ∇


      ∇ FSQROW8
[1]   ⍝ WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
[2]   ⍝ 8/26/81
[3]   ⍝ FURTHER OPTIMIZATION POSSIBLE THROUGH LOOP AVOIDANCE
[4]   ⍝ PROGRAM 'INITIALIZE8' MUST BE RUN BEFORE 'FSQROW8'
[5]   ⍝ PROGRAM 'FSQCOL8' MUST BE RUN AFTER 'FSQROW8'
[6]   ⍝ INPUT VARIABLES H[8×8], F[7×8×8] COME FROM 'INITIALIZE8'
[7]    J←2
[8]   JLP:I←1
[9]   ILP:FT←F[I;;]× 8 8 ⍴H[J;]
[10]   '------------------------'
[11]   'F-SQUARE AND ROW OF H ARE'
[12]   ⎕←I,J
[13]   →ST2×⍳1=∧/0=⎕←(+⌿FT),+/⎕←FT
[14]   →LP,⍴⎕←'COLUMN AND ROW SUMS ARE NOT ALL ZERO'
[15]  ST2:→ST3×⍳1=∧/16=⎕←+/[2]+/[3](F>0)×(⍴F)⍴FT>0
[16]   →LP,⍴⎕←'F-SQUARE, BUT NOT ORTHOGONAL'
[17]  ST3:F←((1-D)\[1] 1 8 8 ⍴FT)+(D←((1↑⍴F)⍴1),0)\[1] F
[18]   '*** ABOVE IS AN ORTHOGONAL F-SQUARE ***'
[19]  LP:→ILP×⍳7≥I←I+1
[20]   →JLP×⍳8≥J←J+1
[21]   '------------------------'
[22]   'END OF F-SQUARE ROW-WISE RUN'
      ∇


      ∇ FSQCOL8
[1]   ⍝ WRITTEN BY S J SCHWAGER, BIOMETRICS UNIT, CORNELL UNIV
[2]   ⍝ 8/26/81
[3]   ⍝ FURTHER OPTIMIZATION POSSIBLE THROUGH LOOP AVOIDANCE
[4]   ⍝ PROGRAM 'INITIALIZE8' MUST BE RUN BEFORE 'FSQROW8'
[5]   ⍝ PROGRAM 'FSQCOL8' MUST BE RUN AFTER 'FSQROW8'
[6]   ⍝ INPUT VARIABLES H[8×8], F[7×8×8] COME FROM 'INITIALIZE8'
[7]    J←2
[8]   JLP:I←1
[9]   ILP:FT←F[I;;]×⍉ 8 8 ⍴H[;J]
[10]   '------------------------'
[11]   'F-SQUARE AND COLUMN OF H ARE'
[12]   ⎕←I,J
[13]   →ST2×⍳1=∧/0=⎕←(+⌿FT),+/⎕←FT
[14]   →LP,⍴⎕←'COLUMN AND ROW SUMS ARE NOT ALL ZERO'
[15]  ST2:→ST3×⍳1=∧/16=⎕←+/[2]+/[3](F>0)×(⍴F)⍴FT>0
[16]   →LP,⍴⎕←'F-SQUARE, BUT NOT ORTHOGONAL'
[17]  ST3:F←((1-D)\[1] 1 8 8 ⍴FT)+(D←((1↑⍴F)⍴1),0)\[1] F
[18]   '*** ABOVE IS AN ORTHOGONAL F-SQUARE ***'
[19]  LP:→ILP×⍳7≥I←I+1
[20]   →JLP×⍳8≥J←J+1
[21]   '------------------------'
[22]   'END OF F-SQUARE COLUMN-WISE RUN'
      ∇
```

References Cited

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